exponents respectively,

$$(-1)^{n-\sigma}(n_1-1)!\,(n_2-1)!\ldots\ldots(n\sigma-1)!\ {}^{s}\lambda_{n_1}\ {}^{s}\lambda_{n_2}\ \ldots\ldots{}^{s}\lambda_{n\sigma},$$

a result that agrees with that obtained in a somewhat different way on page 8 of the German translation of Faa di Bruno's Formes Binaires.

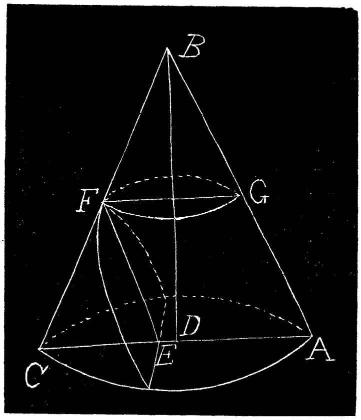
Erlangen, Bavaria, 4 May, 1898.

CONVEX SURFACE AND VOLUME OF CONICAL UNGULÆ.

By G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

Let $BD=c$, $DE=a$, $\tan DBC=n$, $\cot FEC=m$. Also let $DH=h$, $DC=R$, $HF=r$, $EC=d$, then $c=\dfrac{Rh}{R-r}$, $n=\dfrac{R-r}{h}$, $a=R-d$, $m=\dfrac{r-R+d}{h}$.

Then $x^2+z^2=n^2(c-y)^2$, is the equation of the cone, and $x=my+a$, is the equation of the plane.



$$\left(\frac{dz}{dx}\right)^2=\frac{x^2}{z^2},\quad \left(\frac{dz}{dy}\right)^2=\frac{n^4(c-y)^2}{z^2};$$

$$\therefore \sqrt{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2}$$

$$=\frac{n(c-y)\sqrt{(1+n^2)}}{\sqrt{[n^2(c-y)^2-x^2]}}.$$

The limits of x are $my+a=x_2$ and $n(c-y)=x_1$; of y, 0 and $\dfrac{nc-a}{m+n}=y'$.

$$\therefore\ S=2\int_0^{y'}\int_{x_2}^{x_1}\frac{n(c-y)\sqrt{(1+n^2)}}{\sqrt{[n^2(c-y)^2-x^2]}}$$

$$=n\sqrt{(1+n^2)}\int_0^{y'}\left[\pi-2\sin^{-1}\left(\frac{a+my}{n(c-y)}\right)\right](c-y)dy$$

$$=nc^2\sqrt{(1-n^2)}\left[\tfrac{1}{2}\pi-\sin^{-1}\left(\frac{a}{nc}\right)\right]$$

$$-n(a+mc)\sqrt{(1+n^2)}\int_0^{y'}\frac{(c-y)dy}{\sqrt{[n^2(c-y)^2-(a+my)^2]}}.$$

$$n^2(c-y)^2-(a+my)^2=[(mnc+na)^2-(n^2c+am+m^2y-n^2y)^2]/(m^2-n^2).$$

When $m^2>n^2$

$$=(nc+a)(nc-a-2ny).$$

When $m^2=n^2$

$$=[(n^2c+am+m^2y-n^2y)^2-(mnc+na)^2]/(n^2-m^2)$$

When $m^2<n^2$

Let $u=n^2c+am+m^2y-n^2y$.

Then the limits of u are $n^2c+am=u_2$ and $mnc+an=u_1$. When $m^2>n^2$

$$n(a+mc(\sqrt{(1+n^2)}\int_0^{y'}\frac{(c-y)dy}{\sqrt{[n^2(c-y)^2-(a+my)^2]}}$$

$$=\frac{n(a+mc)\sqrt{(1+n^2)}}{\sqrt{[(m^2-n^2)^3]}}\int_{u_2}^{u_1}\frac{(cm^2+am-u)du}{\sqrt{[(an+mcn)^2-u^2]}}$$

$$=\frac{\pi nm(a+mc)^2\sqrt{(1+n^2)}}{2\sqrt{[(m^2-n^2)^3]}}-\frac{nm(a+mc)^2\sqrt{(1+n^2)}}{\sqrt{[(m^2-n^2)^3]}}\sin^{-1}\left(\frac{n^2c+am}{an+mnc}\right)$$

$$-\frac{n(a+mc)\sqrt{(1+n^2)}}{\sqrt{[(m^2-n^2)^3]}}\sqrt{(an+mnc)^2-(n^2c+am)^2}.$$

$$\therefore S=n\sqrt{(1+n^2)}\left[\frac{\pi c^2}{2}-\frac{\pi m(a+mc)^2}{2\sqrt{[(m^2-n^2)^3]}}-c^2\sin^{-1}\left(\frac{a}{nc}\right)\right.$$

$$+\frac{m(a+mc)^2}{\sqrt{[(m^2-n^2)^3]}}\sin^{-1}\left(\frac{n^2c+am}{an+nmc}\right)$$

$$\left.+\frac{(a+mc)}{\sqrt{[(m^2-n^2)^3]}}\sqrt{(an+nmc)^2-(n^2c+am)^2}\right].$$

$$S=\frac{\sqrt{[h^2+(R-r)^2]}}{R-r}\left[\frac{\pi R^2}{2}-\frac{\pi r^2d(r-R+d)}{2\sqrt{[d(d+2r-2R)^3]}}\right.$$

$$+\frac{r^2d(r-R+d)}{\sqrt{[d(d+2r-2R)^3]}}\sin^{-1}\left(\frac{2R-r-d}{r}\right)$$

$$-R^2\sin^{-1}\left(\frac{R-d}{R}\right)+\frac{rd(R-r)}{\sqrt{[d(d+2r-2R)^3]}}\sqrt{r^2-(2R-r-d)^2}.$$

Let $d=2R$. $\therefore S=\frac{\pi\sqrt{[h^2+(R-r)^2]}}{R-r}\left(R^2-\frac{1}{2}\sqrt{Rr}(R+r)\right).$

When $m^2=n^2$, $y'=(nc-a)/2n$.

$$n(a+mc)\sqrt{(1+n^2)}\int_0^{y'}\frac{(c-y)dy}{\sqrt{[n^2(c-y)^2-(a+my)^2]}}$$

$$=n\sqrt{a+nc}\,\sqrt{1+n^2}\int_0^{y'}\frac{(c-y)dy}{\sqrt{[nc-a-2ny]}}=\frac{(2nc+a)\sqrt{[(n^2c^2-a^2)(1+n^2)]}}{3n}.$$

$$\therefore\ S=\sqrt{(1+n^2)}\left[\frac{\pi c^2 n}{2}-nc^2\sin^{-1}\left(\frac{a}{nc}\right)-\frac{(2nc+a)\sqrt{(n^2c^2-a^2)}}{3n}\right].$$

$$\therefore\ S=\frac{\sqrt{[h^2+(R-r)^2]}}{R-r}\left[\frac{\pi R^2}{2}-R^2\sin^{-1}\left(\frac{R-d}{R}\right)-\tfrac{1}{3}(3R-d)\sqrt{d(2R-d)}\right].$$

But $d=2(R-r)$.

$$\therefore\ S=\frac{\sqrt{[h^2+(R-r)^2}}{R-r}\left[\frac{\pi R^2}{2}-R^2\sin^{-1}\left(\frac{2r-R}{R}\right)-\tfrac{2}{3}(R+2r)\sqrt{r(R-r)}\right].$$

When $m^2<n^2$,

$$n(a+mc)\sqrt{(1+n^2)}\int_0^{y'}\frac{(c-y)dy}{\sqrt{[n^2(c-y)^2-(a+my)^2]}}$$

$$=\frac{n(a+mc)\sqrt{(1+n^2)}}{\sqrt{[(n^2-m^2)^3]}}\int_{u_2}^{u_1}\frac{(cm^2+am-u)du}{\sqrt{[u^2-(mnc+an)^2]}}$$

$$=\frac{n(a+mc)\sqrt{(1+n^2)}}{\sqrt{[(n^2-m^2)^3]}}\sqrt{(n^2c+am)^2-(mnc+an)^2}$$

$$-\frac{nm(a+mc)^2\sqrt{(1+n^2)}}{\sqrt{[(n^2-m^2)^3]}}\log\left(\frac{n^2c+am+\sqrt{[(n^2c+am)^2-(mnc+an)^2]}}{mnc+an}\right).$$

$$\therefore\ S=n\sqrt{(1+n^2)}\left[\frac{\pi c^2}{2}-c^2\sin^{-1}\left(\frac{a}{nc}\right)\right.$$

$$-\frac{(a+mc)}{\sqrt{[(n^2-m^2)^3]}}\sqrt{(n^2c+am)^2-(mnc+an)^2}$$

$$\left.+\frac{m(a+mc)^2}{\sqrt{[(n^2-m^2)^3]}}\log\left(\frac{n^2c+am+\sqrt{[(n^2c+am)^2-(mnc+an)^2]}}{mnc+an}\right)\right].$$

$$\therefore\ S=\frac{\sqrt{[h^2+(R-r)^2]}}{R-r}\Big[\frac{\pi R^2}{2}-R^2\sin^{-1}\Big(\frac{R-d}{R}\Big)$$

$$-\frac{rd(R-r)}{\sqrt{[d(2R-d-2r)^3]}}\sqrt{[(2R-r-d)^2-r^2]}$$

$$+\frac{r^2d(r-R+d)}{\sqrt{[d(2R-d-2r)^3]}}\log\Big(\frac{2R-r-d+\sqrt{[(2R-r-d)^2-r^2]}}{r}\Big)\Big].$$

For volume,

$$V=2\int_0^{y'}\int_{x_2}^{x_1}\sqrt{[n^2(c-y)^2-x^2]}dydx$$

$$=\int_0^{y'}\Big[\tfrac{1}{2}\pi\, n^2(c-y)^2-n^2(c-y)^2\sin^{-1}\Big(\frac{a+my}{n(c-y)}\Big)$$

$$-(a+my)\sqrt{[n^2(c-y)^2-(a+my)^2]}\Big]dy=\tfrac{1}{6}\pi n^2c^3-\tfrac{1}{3}n^2c^3\sin^{-1}\Big(\frac{a}{nc}\Big)$$

$$-\tfrac{1}{3}n^2(a+mc)\int_0^{y'}\frac{(c-y)^2dy}{\sqrt{[n^2(c-y)^2-(a+my)^2]}}$$

$$-\int_0^{y'}(a+my)\sqrt{[n^2(c-y)^2-(a+my)^2]}dy.$$

When $m^2>n^2$,

$$\tfrac{1}{3}n^2(a+mc)\int_0^{y'}\frac{(c-y)^2dy}{\sqrt{[n^2(c-y)^2-(a+my)^2]}}$$

$$=\frac{n^2(a+mc)}{3\sqrt{[(m^2-n^2)^5]}}\int_{u_2}^{u_1}\frac{(cm^2+am-u)^2du}{\sqrt{[(an+mcn)^2-u^2]}}$$

$$=\frac{n^2(a+mc)}{6\sqrt{[(m^2-n^2)^5]}}\Big[\frac{\pi(a+mc)^2(2m^2+n^2)}{2}$$

$$-(a+mc)^2(2m^2+n^2)\sin^{-1}\Big(\frac{n^2c+am}{an+nmc}\Big)$$

$$+(n^2c-4cm^2-3am)\sqrt{[(an+nmc)^2-(am+n^2c)^2]}\Big].$$

$$\int_0^{y'}(a+my)\sqrt{[n^2(c-y)^2-(a+my)^2]}dy$$

$$=\frac{1}{\sqrt{[(m^2-n^2)^5]}}\int_{u_2}^{u_1}(mn-an^2-n^2cm)\sqrt{[(an+ncm)^2-u^2]}du$$

$$=\frac{1}{\sqrt{[(m^2-n^2)^5]}}\Big[\tfrac{1}{3}m[(an+cmn)^2-(n^2c+am)]^{\frac{3}{2}}$$

$$+\frac{n^4(a+cm)^3}{2}\sin^{-1}\left(\frac{n^2c+am}{an+nmc}\right)-\frac{\pi n^4(a+cm)^2}{4}$$

$$+\frac{n^2(a+mc)(am+n^2c)}{2}\sqrt{[(an+nmc)^2-(am+n^2c)^2]}\Big].$$

$$\therefore\ V=\tfrac{1}{6}\pi n^2c^3-\tfrac{1}{3}n^2c^3\sin^{-1}(a/nc)-\frac{\pi n^2(a+cm)^3}{6\sqrt{[(m^2-n^2)^3]}}$$

$$+\frac{n^2(a+cm)^3}{3\sqrt{[(m^2-n^2)^3]}}\sin^{-1}\left(\frac{am+n^2c}{an+cmn}\right)$$

$$+\frac{2acn^2+a^2m+n^2c^2m}{3\sqrt{[(m^2-n^2)^3]}}\sqrt{[(an+cnm)^2-(am+n^2c)^2]}.$$

$$\therefore\ V=\frac{R^3h}{3(R-r)}\left[\tfrac{1}{2}\pi-\sin^{-1}\left(\frac{R-d}{R}\right)\right]$$

$$-\frac{hr^3d}{3(R-r)\sqrt{[d(d+2r-2R)^3]}}\left[\tfrac{1}{2}\pi-\sin^{-1}\left(\frac{2R-r-d}{r}\right)\right]$$

$$+\frac{h[2Rd(R-d)+d^2(r-R+d)]}{3\sqrt{[d(d+2r-2R)^3]}}\sqrt{[r^2-(2R-r-d)^2]}.$$

Let $d=2R$. $\therefore\ V=\dfrac{\pi Rh}{3(R-r)}[R^2-r\sqrt{(Rr)}]$.

When $m^2=n^2$, $y'=[(nc-a)/2n]$.

$$\tfrac{1}{3}n^2(a+mc)\int_0^{y'}\frac{(c-y)dy}{\sqrt{[n^2(c-y)^2-(a+my)^2]}}$$

$$=\tfrac{1}{3}n^2\sqrt{[a+nc]}\int_0^{y'}\frac{(c-y)^2dy}{\sqrt{[nc-a-2ny]}}=\frac{\sqrt{(n^2c^2-a^2)}(7n^2c^2+6anc+2a^2)}{45n}.$$

$$\int_0^{y'}(a+my)\sqrt{[n^2(c-y)^2-(a+my)^2]}dy$$

$$=\sqrt{[nc+a]}\int_0^{y'}(a+ny)\sqrt{[nc-a^2-2ny]}dy=\frac{(4a+nc)(nc-a)\sqrt{(n^2c^2-a^2)}}{15n}.$$

$$\therefore\ V=\tfrac{1}{6}\pi n^2c^3-\tfrac{1}{3}n^2c^3\sin^{-1}(a/nc)-\frac{(2n^2c^2+3anc-2a^2)\sqrt{(n^2c^2-a^2)}}{9n}.$$

$$\therefore\ V=\frac{R^3h}{3(R-r)}\left[\tfrac{1}{9}\pi-\sin^{-1}\left(\frac{R-d}{R}\right)\right]-\frac{h(3R^2+Rd-2d^2}{9(R-r)}\sqrt{[d(2R-d)]}.$$

But $d=2R-2r$.

$$\therefore\ V=\frac{h}{3\,(R-r)}\left[\frac{\pi R^3}{2}-R^3\sin^{-1}\left(\frac{2r-R}{R}\right)+\frac{2(3R^2-14Rr+8r^2)}{3}\sqrt{[r(R-r)]}\,\right].$$

When $m^2<n^2$,

$$\tfrac{1}{3}n^2(a+mc)\int_0^{y'}\frac{(c-y)^2\,dy}{\sqrt{[n^2(c-y)^2-(a+my)^2]}}$$

$$=-\frac{n^2(a+mc)}{3\sqrt{[(n^2-m^2)^5]}}\int_{u_2}^{u_1}\frac{(cm^2+am-u)^2\,du}{\sqrt{[u^2-(an+mcn)^2]}}$$

$$=-\frac{n^2(a+mc)}{6\sqrt{[(n^2-m^2)^5]}}\Big[(4cm^2+3am-n^2c)\sqrt{[(n^2c+am)^2-(an+mnc)^2]}$$

$$-(2m^2+n^2)(a+mc)^2\log\left(\frac{n^2c+am+\sqrt{[(n^2c+am)^2-(an+mnc)^2]}}{an+mnc}\right)\Big].$$

$$\int_0^{y'}(a+my)\sqrt{[n^2(c-y)^2-(a+my)^2]}\,dy$$

$$=\frac{1}{\sqrt{[(n^2-m^2)^5]}}\int_{u_2}^{u_1}(mu-an^2-n^2cm)\sqrt{[u^2-(an+mnc)^2]}\,du$$

$$=\frac{1}{\sqrt{[(n^2-m^2)^5]}}\left[\frac{n^2(a+mc)(am+n^2c)}{2}\right]\sqrt{[(n^2c+am)^2-(an+nmc)^2]}$$

$$-\tfrac{1}{3}m[(n^2c+am)^2-(an+nmc)^2]^{\frac{3}{2}}$$

$$-\frac{n^4(a+cm)^3}{2}\log\left(\frac{n^2c+am+\sqrt{[(n^2c+am)^2-(an+nmc)^2]}}{an+nmc}\right).$$

$$\therefore\ V=\tfrac{1}{6}\pi n^2c^3-\tfrac{1}{3}n^2c^3\sin^{-1}\left(\frac{a}{nc}\right)$$

$$+\frac{n^2(a+mc)^3}{3\sqrt{[(n^2-m^2)^3]}}\log\left(\frac{n^2c+am+\sqrt{[(n^2c+am)^2-(an+nmc)^2]}}{an+nmc}\right)$$

$$-\frac{2n^2ac+am^2+n^2c^2m}{3\sqrt{[(n^2-m^2)^3]}}\sqrt{[(n^2c+am)^2-(an+nmc)^2]}.$$

$$\therefore\ V=\frac{R^3h}{3\,(R+r)}\left[\tfrac{1}{2}\pi-\sin^{-1}\left(\frac{R-d}{R}\right)\right]$$

$$+\frac{hr^3d}{3(R-r)\sqrt{[d(2R-d-2r)^3]}}\log\left(\frac{2R-r-d+\sqrt{[(2R-r-d)^2-r^2]}}{r}\right)$$

$$-\frac{h[2Rd(R-d)+d^2(r-R+d)]}{3\sqrt{[d(2R-d-2r)^3]}}\sqrt{[(2R-r-d)^2-r^2}.$$

Let $m=0$.

$$\therefore\ V=\tfrac{1}{3}n^2c^3\left[\tfrac{1}{2}\pi-\sin^{-1}\left(\frac{a}{nc}\right)\right]$$

$$+\frac{a^3}{3n}\log\left(\frac{nc+\sqrt{[n^2c^2-a^2]}}{a}\right)-\tfrac{2}{3}ac\sqrt{[n^2c^2-a^2]}$$

$$=\tfrac{1}{3}c\left[\frac{\pi R^2}{2}-\sin^{-1}\left(\frac{a}{R}\right)+a^3\log\left(\frac{R+\sqrt{[R^2-a^2]}}{a}\right)-2a\sqrt{[R^2-a^2]}\right]$$

$$=\frac{Rh}{3(R-r)}\left[\frac{\pi R^2}{2}-R^2\sin^{-1}(r/R)+\frac{Rr^3}{R-r}\log\left(\frac{R+\sqrt{[R^2-r^2]}}{r}\right)-2r\sqrt{[R^2+r^2]}\right]$$

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

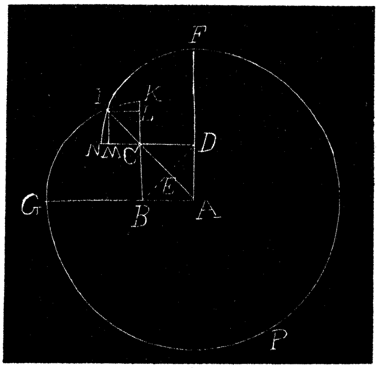
93. Proposed by RAYMOND D. SMITH, Tiffin, Ohio.

A barn 20 feet square is standing in a pasture, and a horse is tied to one corner of it with a rope 50 feet long. Over how much land can he graze ?

I. Solution by B. F. FINKEL, M. Sc., M. A., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Let $ABCD$ be the barn, side $AB=AD=20$ feet; A the corner to which the horse is tied ; and $AF=AG=50$ feet, the length of the rope.

Then $DI=BI=30$ feet ; $AC=DB=20\sqrt{2}$ feet ; $EI=\sqrt{[DI^2-DE^2]}=\sqrt{[30^2-(10\sqrt{2})^2]}$ feet$+10\sqrt{7}$ feet ; $CI=EI-EC=10\sqrt{7}$ feet$-10\sqrt{2}$ feet$=10(\sqrt{7}-\sqrt{2})$ feet ; $CL=CM=\sqrt{[CI^2/2]}=\tfrac{1}{2}CI\sqrt{2}=5(\sqrt{14}-2)$ feet ; $KL=CK-CL=10$ feet$-5(\sqrt{14}-2)$ feet$=5(4-\sqrt{14})$ feet ; and chord $KI=$chord $IN=\sqrt{[KL^2+IL^2]}$.



$\sqrt{[25(4-\sqrt{14})^2+25^2(\sqrt{14}-2)^2]}$ feet $=10\sqrt{[3(4-\sqrt{14}]}$ feet.

2 arc $IK=\tfrac{1}{3}(8$ chord $KI-2IL$*
$=\tfrac{1}{3}\{80\sqrt{[3(4-\sqrt{14})]}-20(\sqrt{7}-\sqrt{2})\}$ feet$=\tfrac{20}{3}\{4\sqrt{[3(4-\sqrt{14})]}-(\sqrt{7}-\sqrt{2})]$ feet.

The area over which the horse can graze$=FAGPF+$sector $FDI+$sector $IBG+$triangle $DCI+$triangle $BCI=FAGPF+2$ sector $FDI+2$ triangle $DCI=FAGPF+2$(quadrant FDN—sector $IDN)+2$ triangle DCI.

But area of $FAGPF=\tfrac{3}{4}\pi AF^2=1875\pi$;

*See *Williamson's Differential Calculus*, pages 64-65, for a proof of this rule. The discovery of this important approximation is due to Huygens. The length of an arc of 30° on a circle of radius 100,000 differs from the true value, assuming $\pi=3.141592$, by about 2 inches. The formula is arc$=\tfrac{1}{3}(8B-A)$ when B is the chord of half the arc and A is chord ofthe arc.